

On the anomaly formula for the Cappell-Miller holomorphic torsion

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Abstract We present an explicit expression of the anomaly formula for the Cappell-Miller holomorphic torsion for Kähler manifolds.

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1 Introduction

As a subsequent paper of the celebrated work [16], Ray and Singer defined a holomorphic torsion associated with the $\bar{\partial}$ -complex of complex manifolds in [17]. In [17, (2.5)], in the flat holomorphic case, they obtained the variation of this holomorphic torsion with respect to the change of Hermitian metrics of the underlying complex manifold, as the constant coefficient of the small time asymptotic expansion of certain trace of the associated heat kernel. Furthermore in [6], using probability method, Bismut, Gillet and Soulé obtained the explicit anomaly formula for the case where the holomorphic bundle is endowed with Hermitian metrics and the base manifold is assumed to be Kähler.

In the recent paper of Cappell and Miller [9], the holomorphic torsion is extended to coupling with an arbitrary holomorphic bundle with compatible connection of type (1,1). However, comparing with the operators dealt with in [6], in the present general setting the associated operators are not necessarily self-adjoint and the torsion is complex valued. Especially, the torsion is independent of the given Hermitian metric of the holomorphic bundle.

In this paper, we get an explicit expression of the anomaly formula for this Cappell-Miller holomorphic torsion for Kähler manifolds. As is obtained in [9], the variation of the holomorphic torsion is the constant term in the Laurent expansion of the supertrace of a certain smooth kernel. To compute the constant term, following [6, Section 1(h)] in spirit, we introduce the Grassmann variables $da, d\bar{a}$ and identify the constant term as the coefficient of $dad\bar{a}$ of $\lim_{t \rightarrow 0^+} \text{Tr}_s[\exp(-tI'_t)]$, where for any $t > 0$, I'_t is a generalized non-self-adjoint Laplacian with parameters $da, d\bar{a}$.

In our final calculation, we do not use probability theory as in [6], but modify the proof which is presented in [1, Chapter 4] to deduce the local index theorem [1, Theorem 4.1]. The technical difficulties

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in the modification are the singular terms turning up in the rescaled operator and the convergence of the rescaled heat kernel for the introduced rescaling parameter ε as $\varepsilon \rightarrow 0$. We first modify the estimates on heat kernels given in [1, Chapter 2], which essentially depend on the properties of the introduced Grassmann variables, then using Donnelly’s conjugation technique and the general Mehler formula for a generalized harmonic oscillator (cf. [10]), we overcome the difficulties.

As the referees pointed out to us, Bismut and Köhler [7] established the anomaly formula for the holomorphic torsion forms and Ma [14] extended it to the equivariant case. Modifying the proof of [14, Theorem 2.13], one can work out the anomaly formula for Cappell-Miller holomorphic torsion directly. We shall explain it in Remark 4.9.

The rest of this paper is organized as follows. In Section 2, we recall the basic definition of the Cappell-Miller torsion of the holomorphic bundle endowed with a compatible (1,1) connection over complex manifolds and state the anomaly formula for the case where the base manifold assumed to be Kähler. In Section 3, we deduce some estimates on the heat kernels in a little more general case than what we need. In Section 4, using the local index theorem techniques, we prove the anomaly formula stated in Section 2.

2 Cappell-Miller torsion of the holomorphic bundle with a compatible (1,1) connection

In this section, we recall the definition of the Cappell-Miller torsion of the holomorphic bundle with a compatible (1,1) connection and state an anomaly formula for the Cappell-Miller torsion under the Kähler condition.

2.1 The Cappell-Miller holomorphic torsion

Let (M, J) be a complex manifold with complex structure J and its complex dimension be n . Let TM be the corresponding real tangent bundle. Let g^{TM} be any Riemannian metric on TM compatible with J and ∇^{TM} be the corresponding Levi-Civita connection.

Let $E \rightarrow M$ be a complex holomorphic bundle over M endowed with a connection ∇^E . Let g^E be a Hermitian metric on E .

For $0 \leq r \leq 2n$, let $\Omega^r(M, E) = \Gamma(M, \Lambda^r(T^*M) \otimes E)$ be the space of smooth r -forms on M with values in E .

The complex structure J induces a splitting $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$, where $T^{(1,0)}M$ and $T^{(0,1)}M$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Let $T^{*(1,0)}M$ and $T^{*(0,1)}M$ be the corresponding dual bundles.

For any $0 \leq p, q \leq n$, let

$$\Omega^{p,q}(M, E) = \Gamma(M, \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M) \otimes E)$$

be the space of smooth (p, q) -forms on M with values in E . Set

$$\Omega^{*,*}(M, E) = \bigoplus_{p,q=0}^n \Omega^{p,q}(M, E).$$

We have the direct sum decomposition $\Omega^r(M, \mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}(M, \mathbb{C})$ and the differentials $d : \Omega^r(M, \mathbb{C}) \rightarrow \Omega^{r+1}(M, \mathbb{C})$, $\partial : \Omega^{p,q}(M, \mathbb{C}) \rightarrow \Omega^{p+1,q}(M, \mathbb{C})$ and $\bar{\partial} : \Omega^{p,q}(M, \mathbb{C}) \rightarrow \Omega^{p,q+1}(M, \mathbb{C})$ with $d = \partial + \bar{\partial}$.

We will denote by $\langle \cdot, \cdot \rangle$ the \mathbb{C} -bilinear form on $TM \otimes_{\mathbb{R}} \mathbb{C}$ induced by g^{TM} . $T^{(1,0)}M$ is a holomorphic vector bundle with Hermitian metric induced by g^{TM} . Let $\{e_{2i-1}\}_{i=1}^n \cup \{e_{2i} := J e_{2i-1}\}_{i=1}^n$ be an orthonormal frame of TM and $\{e^i\}_{i=1}^{2n}$ be its dual frame. Then

$$\omega_j = \frac{1}{\sqrt{2}}(e_{2j-1} - \sqrt{-1} e_{2j}), \quad j = 1, \dots, n \tag{1}$$

form a local orthonormal frame of $T^{(1,0)}M$ with dual frame $\{\omega^j\}_{j=1}^n$ (cf. [15, (1.2.34)]). We fix this notation throughout the remaining part and use it without further notice.

Let Θ be the real (1,1)-form defined by

$$\Theta(u, v) = g^{TM}(Ju, v), \quad \text{for any } u, v \in \Gamma(TM). \tag{2}$$

We call Θ as in (2) a Kähler form on M . The metric g^{TM} on TM is called a Kähler metric and the complex manifold (M, J) is called a Kähler manifold if Θ is a closed form (cf. [15, Definition 1.2.7]).

There is a natural Hermitian metric on $\Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M) \otimes E$ induced by g^{TM} and g^E , which we denote by $\langle \cdot, \cdot \rangle_{\Lambda^{*,*} \otimes E}$. By Wirtinger Theorem (cf. [13, pp. 31]), we know that the volume form of M determined by g^{TM} is given by $\frac{\Theta^n}{n!}$. Therefore, the L^2 -scalar product on $\Omega^{*,*}(M, E)$ is given by

$$\langle\langle \alpha, \beta \rangle\rangle := \int_M \langle \alpha, \beta \rangle_{\Lambda^{*,*} \otimes E} \frac{\Theta^n}{n!}, \quad \text{for any } \alpha, \beta \in \Omega^{*,*}(M, E). \tag{3}$$

The complex Hodge star operator is a complex conjugate linear mapping

$$* : \Omega^{p,q}(M, \mathbb{C}) \rightarrow \Omega^{n-p, n-q}(M, \mathbb{C})$$

such that if $\alpha, \beta \in \Omega^{*,*}(M, \mathbb{C})$, then (cf. [13, pp. 80–82])

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle_{\Lambda^{*,*}} \frac{\Theta^n}{n!}. \tag{4}$$

The formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ with respect to the L^2 -scalar product is given by

$$\bar{\partial}^* = - * \bar{\partial} * . \tag{5}$$

Let conj be the natural conjugate mapping induced by the bundle automorphism (cf. [9, p. 141])

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T^*M \otimes_{\mathbb{R}} \mathbb{C}, \quad v \otimes \lambda \mapsto v \otimes \bar{\lambda} \quad \text{for any } v \in T^*M, \lambda \in \mathbb{C}. \tag{6}$$

Then $\hat{*} := \text{conj} *$ is a complex linear mapping. Clearly, $\hat{*} = \text{conj} * = * \text{conj}$.

We now introduce the Clifford algebra (cf. [15, Section 1.3.1]).

For any $v \in TM$ with decomposition $v = v^{(1,0)} + v^{(0,1)} \in T^{(1,0)}M \oplus T^{(0,1)}M$, let $\bar{v}^{(1,0),*} \in T^{*(0,1)}M$ be the metric dual of $v^{(1,0)}$. The Clifford action of v on the bundle $\Lambda(T^{*(0,1)}M) = \Lambda^{\text{even}}(T^{*(0,1)}M) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}M)$ is defined by

$$c(v) = \sqrt{2} (\bar{v}^{(1,0),*} \wedge -i_{v^{(0,1)}}), \tag{7}$$

where \wedge and i denote the exterior and interior multiplications, respectively. We verify easily that for $U, V \in TM$,

$$c(U)c(V) + c(V)c(U) = -2\langle U, V \rangle. \tag{8}$$

We identify TM with T^*M by the given metric g^{TM} , and sometimes write $c(e^i)$ as $c(e_i)$. In the sequel, we do not distinguish between $c(e^i)$ and $c(e_i)$.

Since E is holomorphic, the usual $\bar{\partial}$ operator on $\Omega^{*,*}(M, \mathbb{C})$ has a unique natural extension to $\Omega^{*,*}(M, E)$, $\bar{\partial}_E : \Omega^{p,q}(M, E) \rightarrow \Omega^{p,q+1}(M, E)$ (cf. [9, Section 3]).

Under the splitting $\Omega^1(M, E) = \Omega^{1,0}(M, E) \oplus \Omega^{0,1}(M, E)$, the connection ∇^E decomposes as sum: $\nabla^E = (\nabla^E)^{1,0} \oplus (\nabla^E)^{0,1}$ with

$$(\nabla^E)^{1,0} : \Gamma(M, E) \rightarrow \Omega^{1,0}(M, E), \quad (\nabla^E)^{0,1} : \Gamma(M, E) \rightarrow \Omega^{0,1}(M, E). \tag{9}$$

Moreover, if we extend ∇^E on $\Gamma(M, E)$ in a unique way to $\Omega^{*,*}(M, E)$ by Leibniz formula (cf. [1, p. 21]), then the extended ∇^E splits into two pieces $\nabla^E = (\nabla^E)^{1,0} + (\nabla^E)^{0,1}$, which also satisfy the Leibniz formula (cf. [1, p. 131]).

Definition 2.1 [9]. *The connection ∇^E is said to be compatible with the holomorphic structure on E if $(\nabla^E)^{0,1} = \bar{\partial}_E$. The connection ∇^E is said to be of type (1, 1) if the curvature $(\nabla^E)^2$ is of type (1, 1).*

In the sequel, we fix any $p, 0 \leq p \leq n$, and set $\Omega^{p,*}(M, E) = \bigoplus_{q=0}^n \Omega^{p,q}(M, E)$. Let ∇^E be a compatible (1,1) connection. Following [9, Section 3], we define

$$\bar{\partial}_{E,(\nabla^E)^{1,0}}^* = -(\widehat{*} \otimes 1)(\nabla^E)^{1,0}(\widehat{*} \otimes 1), \tag{10}$$

and

$$\square_{E,\bar{\partial}} = \bar{\partial}_E \bar{\partial}_{E,(\nabla^E)^{1,0}}^* + \bar{\partial}_{E,(\nabla^E)^{1,0}}^* \bar{\partial}_E. \tag{11}$$

Since $\widehat{*}^2 = *^2 = (-1)^{p+q}$ on $\Omega^{p,q}(M, \mathbb{C})$, we get

$$(\bar{\partial}_{E,(\nabla^E)^{1,0}}^*)^2 = (-1)^{p+q+1}(\widehat{*} \otimes 1) ((\nabla^E)^{1,0})^2 (\widehat{*} \otimes 1) = 0. \tag{12}$$

Using the following identities

$$(\nabla^E)^{1,0} = \partial \otimes 1 + \omega^j \wedge \nabla_{\omega_j}^E, \quad (\nabla^E)^{0,1} = \bar{\partial} \otimes 1 + \bar{\omega}^j \wedge \nabla_{\bar{\omega}_j}^E, \tag{13}$$

we get

$$\bar{\partial}_{E,(\nabla^E)^{1,0}}^* = \bar{\partial}^* \otimes 1 - i\bar{\omega}_j \nabla_{\omega_j}^E, \quad \bar{\partial}_E = \bar{\partial} \otimes 1 + \bar{\omega}^j \wedge \nabla_{\bar{\omega}_j}^E. \tag{14}$$

Set

$$D = \sqrt{2}(\bar{\partial}_E + \bar{\partial}_{E,(\nabla^E)^{1,0}}^*). \tag{15}$$

Then from (11), (14), (15), we deduce that

$$D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) \otimes 1 + c(e_i) \otimes \nabla_{e_i}^E, \quad D^2 = 2\square_{E,\bar{\partial}}. \tag{16}$$

Denote $\mathcal{E} = \Lambda^p(T^{*(1,0)}M) \otimes E$. If we assume in addition that (M, J, g^{TM}) is a Kähler manifold, then using [1, Proposition 3.67], [15, Lemma 1.4.4], the operator D acting on $\Omega^{p,*}(M, E) = \Omega^{0,*}(M, \mathcal{E})$ can be specified as follows.

Proposition 2.2. *D can be regarded as a Dirac operator on the Clifford module $\Lambda(T^{*(0,1)}M) \otimes \mathcal{E}$ associated with the Clifford connection*

$$\nabla^{\text{Cl} \otimes \mathcal{E}} = \nabla^{\text{Cl}} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}}, \quad i.e. \quad D = c(e_i) \nabla_{e_i}^{\text{Cl} \otimes \mathcal{E}}, \tag{17}$$

where ∇^{Cl} is the natural connection on $\Lambda(T^{*(0,1)}M)$ induced by the Levi-Civita connection ∇^{TM} (cf. [1, p. 28]) and $\nabla^{\mathcal{E}}$ is the natural tensor connection induced by ∇^{TM} and ∇^E .

Let $\text{Spec}(\square_{E,\bar{\partial}})$ be the spectrum of $\square_{E,\bar{\partial}}$ acting on $\Omega^{0,*}(M, \mathcal{E})$. Denote

$$\text{Re}(\text{Spec}(\square_{E,\bar{\partial}})) = \{\text{Re}(\lambda) \mid \lambda \in \text{Spec}(\square_{E,\bar{\partial}})\}.$$

Pick any $a > 0$ such that $a \notin \text{Re}(\text{Spec}(\square_{E,\bar{\partial}}))$. Set

$$\Omega_{<a}^{p,*}(M, E) = \bigoplus_{\text{Re } \lambda < a} \Omega_{\{\lambda\}}^{p,*}(M, E).$$

Let $\Pi_a : \Omega^{p,*}(M, E) \rightarrow \Omega_{<a}^{p,*}(M, E)$ be the orthogonal projection with respect to the natural L^2 -scalar product (cf. (3)) on $\Omega^{0,*}(M, \mathcal{E})$. Set $P_a = 1 - \Pi_a$.

We recall some facts obtained in [9, Section 4].

$(\Omega_{<a}^{p,*}(M, E), \bar{\partial}_E)$ forms a finite-dimensional complex. Moreover, the inclusion of the complexes

$$(\Omega_{<a}^{p,*}(M, E), \bar{\partial}_E) \subset (\Omega^{p,*}(M, E), \bar{\partial}_E)$$

induces an isomorphism on cohomology. That is,

$$H^q(\Omega_{<a}^{p,*}(M, E), \bar{\partial}_E) \xrightarrow{\cong} H_{\bar{\partial}}^{p,q}(M, E). \tag{18}$$

$\widehat{*} \otimes 1$ induces a \mathbb{C} -linear isomorphism of the complex $(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_{E,(\nabla^E)_{1,0}}^*)$ to the complex $(\Omega_{<a}^{n-*,n-p}(M, E), (-1)^{p+1+*}(\nabla^E)^{1,0})$. We have isomorphisms

$$\begin{aligned} H_{(\nabla^E)_{1,0}}^{n-q,n-p}(M, E) &\xrightarrow{\cong} H^{n-q}(\Omega_{<a}^{*,n-p}(M, E), (-1)^{p+1+*}(\nabla^E)^{1,0}) \\ &\xrightarrow{\widehat{*} \otimes 1} H_q(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_{E,(\nabla^E)_{1,0}}^*). \end{aligned} \tag{19}$$

From [9, Section 6], we know that there is a natural non-vanishing algebraic torsion invariant associated with the complex $(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_E, \overline{\partial}_{E,(\nabla^E)_{1,0}}^*)$,

$$\begin{aligned} &\text{torsion}(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_E, \overline{\partial}_{E,(\nabla^E)_{1,0}}^*) \\ &\in \det(H^*(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_E)) \otimes \det(H_*(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_{E,(\nabla^E)_{1,0}}^*))^{-1}. \end{aligned}$$

Using the isomorphisms (18) and (19), we regard it as an element of the complex line

$$\det(H_{\overline{\partial}}^{p,*}(M, E)) \otimes \det(H_{(\nabla^E)_{1,0}}^{n-*,n-p}(M, E))^{-1}.$$

Note here the complex line is independent of the metric g^{TM} .

Let N be the number operator, i.e., N acts on $\Omega^{0,q}(M, \mathcal{E})$ by multiplication by q . By [9, Section 11], for $\text{Re}(s) > \frac{n}{2}$, the following zeta function is well defined,

$$\zeta_a(s) = \text{Tr}_s[N(\square_{E,\overline{\partial}})^{-s} P_a]. \tag{20}$$

Moreover, $\zeta_a(s)$ has a meromorphic extension to the whole complex plane and is analytic at $s = 0$. Consequently, the derivative at $s = 0$, $\zeta'_a(0)$ is meaningful.

By [9, Lemma 4.3] and [9, (4.8)], we know that the combination

$$\text{torsion}(\Omega_{<a}^{p,*}(M, E), \overline{\partial}_E, \overline{\partial}_{E,(\nabla^E)_{1,0}}^*) \cdot \exp(\zeta'_a(0)) \tag{21}$$

is independent of the choice of $a > 0$ with $a \notin \text{Re}(\text{Spec}(\square_{E,\overline{\partial}}))$ and the Hermitian metric g^E .

Definition 2.3 (cf. [9, p. 152]). *The non-vanishing element of the complex line*

$$\det(H_{\overline{\partial}}^{p,*}(M, E)) \otimes \det(H_{(\nabla^E)_{1,0}}^{n-*,n-p}(M, E))^{-1}$$

defined in (21) is called the Cappell-Miller holomorphic torsion and is denoted by $\tau_{\text{holo},p}(M, E)$.

2.2 An anomaly formula for the Cappell-Miller holomorphic torsion for Kähler manifolds

We indicate the characteristic classes which we will use.

Let g^{TM} be a Kähler metric on TM . Let R^+ be the curvature of $T^{(1,0)}M$ with the natural connection induced by the Levi-Civita connection ∇^{TM} associated with g^{TM} . Let $R^E = (\nabla^E)^2$ be the curvature of ∇^E . If $\mathcal{A} \in \text{End}(T^{(1,0)}M)$, set

$$\begin{aligned} \text{Td}(\mathcal{A}) &= \det_{T^{(1,0)}M} \left(\frac{\mathcal{A}}{e^{\mathcal{A}} - 1} \right), \quad \det(I + t\mathcal{A}) = 1 + t\sigma_1(\mathcal{A}) + \cdots + t^n \sigma_n(\mathcal{A}), \\ \text{Td}_p(\mathcal{A}) &= \text{Td}(\mathcal{A}) \sigma_p(\exp \mathcal{A}). \end{aligned} \tag{22}$$

Definition 2.4. *Set*

$$\text{Td}_p(T^{(1,0)}M, g^{TM}) = \text{Td}_p \left(\frac{R^+}{2\pi\sqrt{-1}} \right), \quad \text{ch}(E, \nabla^E) = \text{Tr} \left[\exp \left(\frac{-R^E}{2\pi\sqrt{-1}} \right) \right]. \tag{23}$$

As in [4, p. 58], let $P = \bigoplus_{j=0}^n \Omega^{j,j}(M, \mathbb{C})$. Let $P' \subset P$ be the set of smooth forms $\alpha \in P$ such that there exist smooth forms β, γ for which $\alpha = \partial\beta + \overline{\partial}\gamma$. When $\alpha, \alpha' \in P$, we write $\alpha \equiv \alpha'$ if $\alpha - \alpha' \in P'$. Then the pairing of the elements of P/P' with the element of P which is closed and has compact support is well defined.

Let g'^{TM} be another Kähler metric on TM .

By the results of [4, Section (f)], there is uniquely defined Bott-Chern class

$$\widetilde{\text{Td}}_p(T^{(1,0)}M, g^{TM}, g'^{TM}) \in P/P'$$

such that

$$\frac{\bar{\partial}\partial}{2\pi\sqrt{-1}}\widetilde{\text{Td}}_p(T^{(1,0)}M, g^{TM}, g'^{TM}) = \text{Td}_p(T^{(1,0)}M, g^{TM}) - \text{Td}_p(T^{(1,0)}M, g'^{TM}). \tag{24}$$

Let $\tau_{\text{holo},p}, \tau'_{\text{holo},p}$ be the Cappell-Miller holomorphic torsions associated with the Kähler metrics g^{TM}, g'^{TM} , respectively.

Using the above notations, we state our main theorem as follows.

Theorem 2.5 (Compare with [6, Theorem 1.23]). *The following identity holds*

$$\frac{\tau'_{\text{holo},p}}{\tau_{\text{holo},p}} = \exp\left(\int_M \widetilde{\text{Td}}_p(T^{(1,0)}M, g^{TM}, g'^{TM}) \cdot \text{ch}(E, \nabla^E)\right). \tag{25}$$

3 Some estimates on heat kernels

In this section, we deduce some estimates on heat kernels, which will be needed in the next section.

Throughout this section, we assume that M is a compact oriented smooth manifold of dimension n with a Riemannian metric g^{TM} . Let e_1, \dots, e_n be a local orthonormal frame of TM with respect to g^{TM} . Let $C(M)$ be the Clifford bundle associated with g^{TM} (cf. [1, Definition 3.30]). For any $0 \leq i \leq n$, denote

$$\Omega^i(M) = \Gamma(\Lambda^i(T^*M)) \quad \text{and} \quad \Omega^*(M) = \bigoplus_{i=0}^n \Omega^i(M). \tag{26}$$

Let \mathcal{F} be a real vector bundle of dimension m over M with a connection $\nabla^{\mathcal{F}}$.

This section is self-contained.

3.1 The special heat kernel depending on parameters

Let $\vartheta_1, \dots, \vartheta_i$ be auxiliary Grassmann variables, and any of which anticommutes with $C(X)$, where $C(X)$ is the element $X \in T_xM$ considered as an element of $C(M)$. Assume also that the multiplication of any $q + 1$ variables of the above given Grassmann variables vanishes, where q is some fixed integer.

Let $R(\vartheta_1, \dots, \vartheta_i)$ be the Grassmann algebra generated by $1, \vartheta_1, \dots, \vartheta_i$ (cf. [3]). If $\omega \in R(\vartheta_1, \dots, \vartheta_i)$, then ω is a linear combinations of $\vartheta_{i_1} \cdots \vartheta_{i_k}$, where $1 \leq i_1 < \dots < i_k \leq i$. We say the monomial $\vartheta_{i_1} \cdots \vartheta_{i_k}$ is of degree k . Clearly, $k \leq q$.

For any $t > 0$, there is a homomorphism of algebras

$$\psi_t : R(\vartheta_1, \dots, \vartheta_i) \longrightarrow R(\vartheta_1, \dots, \vartheta_i), \tag{27}$$

which for $1 \leq j \leq i$, maps ϑ_j in $\frac{\vartheta_j}{\sqrt{t}}$ (cf. [8, Section 4]).

Defining the elements of $\Omega^*(M) \otimes \text{End}(\mathcal{F})$ to be of degree zero, we give every monomial of $\Omega^*(M) \otimes \text{End}(\mathcal{F}) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$, say $\varphi_{i_1 \dots i_k} \vartheta_{i_1} \cdots \vartheta_{i_k}$, where $\varphi_{i_1 \dots i_k} \in \Omega^*(M) \otimes \text{End}(\mathcal{F})$, a natural degree. The homomorphism in (27) can be extended to $\Omega^*(M) \otimes \text{End}(\mathcal{F}) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$ in an obvious way.

For any $t > 0$, $\omega \in \Omega^1(M) \otimes \text{End}(\mathcal{F}) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$, set

$${}^t\omega = \psi_t(\omega) \quad \text{and} \quad {}^t\nabla^{\mathcal{F}} = \nabla^{\mathcal{F}} + {}^t\omega. \tag{28}$$

We may assume that ${}^t\omega$ has no degree 0 term¹⁾.

¹⁾ In fact, let ${}^t\omega^{\{0\}}$ be the degree 0 term of ${}^t\omega$, then $\widetilde{\nabla}^{\mathcal{F}} := {}^t\omega^{\{0\}} + \nabla^{\mathcal{F}}$ is a connection on \mathcal{F} . Replacing $\nabla^{\mathcal{F}}$ by $\widetilde{\nabla}^{\mathcal{F}}$, then ${}^t\omega - {}^t\omega^{\{0\}}$ has no degree 0 term.

For any $t > 0$, $\rho \in \text{End}(\mathcal{F}) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$, set ${}^t\rho = \psi_t(\rho)$. We now consider the smooth family of differential operators

$$I'_t = -(\nabla_{e_i}^{\mathcal{F}} + {}^t\omega(e_i))^2 + {}^t\rho \quad \text{and} \quad I_t = tI'_t.$$

Here we use the same notation as in [2, Section 3(b)],

$$(\nabla_{e_i}^{\mathcal{F}} + {}^t\omega(e_i))^2 := \sum_{i=1}^n (\nabla_{e_i}^{\mathcal{F}} + {}^t\omega(e_i))^2 - \nabla_{\nabla_{e_i}^{\mathcal{F}} e_i}^{\mathcal{F}} - {}^t\omega(\nabla_{e_i}^{TM} e_i). \tag{29}$$

For any $t > 0$, let $k(x, y, s, t)$ be the smooth section of the bundle $\mathcal{F} \boxtimes \mathcal{F}^*$ over $\mathbb{R}_+ \times M \times M$, satisfying the following equation with boundary condition at $s = 0$:

$$\begin{cases} (\partial_s + I_t)k(x, y, s, t) = 0, \\ \lim_{s \rightarrow 0^+} \int_{y \in M} k(x, y, s, t)l(y)dy = l(x). \end{cases} \tag{30}$$

Here we use the same convention as in [1, pp. 72] that $\mathcal{F} \boxtimes \mathcal{F}^* = \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{F}^*$, where π_1, π_2 are the projections from $M \times M$ onto the first and second factor M respectively.

Set $u = st$, then $\partial_s = t\partial_u$. The equation in (30) is equivalent to

$$\begin{cases} (\partial_u + I'_t)\tilde{k}(x, y, u, t) = 0, \\ \lim_{u \rightarrow 0^+} \int_{y \in M} \tilde{k}(x, y, u, t)l(y)dy = l(x), \end{cases} \tag{31}$$

where $\tilde{k}(x, y, u, t) = k(x, y, s, t)$. In (30) and (31), $l(x)$ is any smooth section of \mathcal{F} and the limit is meant in the uniform norm $\|l\|_0 = \sup_{x \in M} \|l(x)\|$ for any metric on \mathcal{F} . I'_t is a generalized Laplacian with parameters in $R(\vartheta_1, \dots, \vartheta_i)$ for any $t > 0$.

Let x and y be sufficiently close points in M . Let l_1, \dots, l_m be a local frame of \mathcal{F} on a neighborhood of y , which are parallel along the radical geodesic curve $x_s = \exp_y s\mathbf{x} : [0, 1] \mapsto M$, with respect to the connection $\nabla^{\mathcal{F}}$.

For any $t > 0$, we define $\tau^t(x_s, y) \in \text{Hom}(\mathcal{F}_y, \mathcal{F}_{x_s})$ as follows

$$\tau^t(x_s, y)(l_1(y), \dots, l_m(y)) = (l_1(x_s), \dots, l_m(x_s))A(s), \tag{32}$$

where

$$A(s) = I + \sum_{k=1}^q (-1)^k \int_{s\Delta_k} {}^t\omega(t_k) \cdots {}^t\omega(t_1) dt_1 \cdots dt_k, \tag{33}$$

and $s\Delta_k, k \geq 1$, is a rescaled simplex given by

$$\{(t_1, \dots, t_k) | 0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq s\}. \tag{34}$$

Checking directly, we know that $A(s)$ is non-degenerated and that

$${}^t\nabla_{\partial_s}^{\mathcal{F}}(\tau^t(x_s, y)l_k(y)) = 0, \quad \text{for } k = 1, \dots, m. \tag{35}$$

Therefore, it gives another smooth trivialization of \mathcal{F} over the neighborhood of y corresponding to ${}^t\nabla^{\mathcal{F}}$.

Let q_u be the smooth kernel modeled on the Euclidean heat kernel: in a normal coordinate around y ($x = \exp_y \mathbf{x}$),

$$q_u(x, y) = (4\pi u)^{-\frac{n}{2}} \exp\left(-\frac{\|\mathbf{x}\|^2}{4u}\right).$$

We fix $y \in M$ and write q_u for the section $x \mapsto q_u(x, y)$.

Set $j(\mathbf{x}) = \det^{\frac{1}{2}}(g_{ij}(\mathbf{x}))$. Using [1, Theorem 2.26], in the above new trivialization, the formal solution²⁾ of the heat equation (31) is given by

$$k_u(x, y; t)^{\text{formal}} = q_u(x, y) \sum_{i=0}^{\infty} u^i \Phi_i(x, y; t), \tag{36}$$

where $\Phi_0(x, y; t) = j^{-\frac{1}{2}}(\mathbf{x})\tau^t(x, y)$, and for $i > 0$,

$$\Phi_i(x, y; t) = -j^{-\frac{1}{2}}(\mathbf{x})\tau^t(x, y) \int_0^1 s^{i-1} j^{\frac{1}{2}}(s\mathbf{x})\tau^t(x_s, y)^{-1} (I'_{t,x} \Phi_{i-1})(x_s, y; t) ds.$$

Set

$$\psi(s) = \begin{cases} 1, & \text{if } s < \frac{\varepsilon^2}{4}, \\ 0, & \text{if } s > \varepsilon^2, \end{cases}$$

where ε is chosen to be smaller than the injectivity radius of the manifold M .

For N large enough, we define the approximate solution by the formula

$$k_u^N(x, y; t) = \psi(d(x, y)^2) q_u(x, y) \sum_{i=0}^N u^i \Phi_i(x, y; t). \tag{37}$$

In the sequel, we fix N large enough. In a neighborhood of the diagonal, we write $y = \exp_x \mathbf{y}$, with $\mathbf{y} \in T_x M$, and identify \mathcal{F}_x to \mathcal{F}_y by parallel transport along the geodesic joining x and y with respect to $\nabla^{\mathcal{F}}$. That is, if l is a smooth section of \mathcal{F} , we denote by $l(x, \mathbf{y})$ the function of \mathbf{y} such that $l(x, \mathbf{y}) = \tau(x, y)l(y) \in \mathcal{F}_x$; if Φ is a section of $\mathcal{F} \boxtimes \mathcal{F}^*$, we denote by $\Phi(x, \mathbf{y})$ an endomorphism of \mathcal{F}_x such that $\Phi(x, \mathbf{y}) = \Phi(x, y)\tau(x, y)^{-1}$. Let $\Psi_i(x, \mathbf{y}; t) = \psi(\|\mathbf{y}\|^2)\Phi_i(x, \mathbf{y}; t)$. We have $\Psi_0(x, 0) = \text{Id}_{\mathcal{F}_x}$.

The explicit formulas for $\Phi_i(x, y; t)$ and (33) imply that $\Psi_i(x, \mathbf{y}; t)$ can be expressed as

$$\Psi_i(x, \mathbf{y}; t) = \sum_{j=0}^q t^{-\frac{j}{2}} \phi_{ij}(x, \mathbf{y}), \quad \text{for all } i \geq 0, \tag{38}$$

where $\phi_{ij}(x, \mathbf{y}) \in \text{End}(\mathcal{F}_x) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$.

Thus, the approximate solution can be expressed as

$$k_u^N(x, y; t) = q_{st}(x, y) \sum_{i=0}^N (st)^i \sum_{j=0}^q t^{-\frac{j}{2}} \phi_{ij}(x, \mathbf{y}) \tau(x, y). \tag{39}$$

Let $(\mathcal{B}, \|\cdot\|)$ be a normed space. We introduced a norm $\|\cdot\|_{\mathcal{B} \otimes R}$ on $\mathcal{B} \otimes R(\vartheta_1, \dots, \vartheta_i)$ as follows. For

$$\varphi = \sum_{\substack{1 \leq k \leq q \\ 1 \leq i_1 < \dots < i_k \leq i}} \varphi_{i_1 \dots i_k} \vartheta_{i_1} \cdots \vartheta_{i_k} \in \mathcal{B} \otimes R(\vartheta_1, \dots, \vartheta_i),$$

we define

$$\|\varphi\|_{\mathcal{B} \otimes R} = \max_{\substack{1 \leq k \leq q \\ 1 \leq i_1 < \dots < i_k \leq i}} \|\varphi_{i_1 \dots i_k}\|. \tag{40}$$

Proceeding as [1, Section 2.5], we deduce that there exists a unique smooth solution to (30), which is called the heat kernel and is denoted by $p_u(x, y; t)$ (cf. [1, Proposition 2.17 and Theorem 2.30]). Therefore, the unique smooth kernel $k(x, y, s, t)$ determined by (30) equals $p_u(x, y; t)|_{u=st}$. We can get the following estimate.

Theorem 3.1 (Compare with [1, Theorem 2.23]). *Assume $0 < s, t < T$. Then there exists a constant $C > 0$, such that*

$$\|\partial_s^k (p_u(x, y; t) - k_u^N(x, y; t))\|_{\ell} \leq C \cdot s^{N - \frac{n}{2} - \frac{\ell}{2} - k + 1} \cdot t^{N - \frac{k+2}{2}q - \frac{n}{2} - \frac{\ell}{2} + 1}, \tag{41}$$

where $\|\cdot\|_{\ell}$ is defined using the usual \mathcal{C}^{ℓ} -norm (cf. [1, p. 71]) on \mathcal{C}^{ℓ} -sections of $\mathcal{F} \boxtimes \mathcal{F}^*$ over $\mathbb{R}_+ \times M \times M$ as in (40).

²⁾ Because the manifold considered here is oriented, there is a little but not essential difference in the expression of the formal solution.

3.2 The rescaled heat kernel and the Mehler formula

We continue the discussion of the last subsection.

Take any $x_0 \in M$ and trivialize the vector bundle \mathcal{F} in a neighborhood of x_0 by parallel transport along radical geodesics with respect to $\nabla^{\mathcal{F}}$ (cf. [1, pp. 153–154]). More precisely, let $V = T_{x_0}M$, $F = \mathcal{F}_{x_0}$ and $U = \{\mathbf{x} \in V \mid \|\mathbf{x}\| < \varepsilon\}$, where ε is the injectivity radius of the compact manifold M at x_0 . We identify U by means of exponential map $\mathbf{x} \mapsto \exp_{x_0}\mathbf{x}$ with a neighborhood of x_0 in M . For $x = \exp_{x_0}\mathbf{x}$, the fibre \mathcal{F}_x and F are identified by the parallel transport map $\tau(x_0, x) : \mathcal{F}_x \rightarrow F$ along the geodesic $x_s = \exp_{x_0}s\mathbf{x}$.

Choose an orthonormal basis ∂_i of $V = T_{x_0}M$, with dual basis dx^i of $T_{x_0}^*M$, and let $c^i = c(dx^i) \in \text{End}(E)$. Let S be the spinor space of V^* and let $W = \text{Hom}_{C(V^*)}(S, F)$ be the auxiliary vector space such that $F = S \otimes W$, so that $\text{End}(F) \cong \text{End}(S) \otimes \text{End}(W) \cong C(V^*) \otimes \text{End}(W)$. Let e_i be the local orthonormal frame obtained by parallel transports along the geodesics from the orthonormal basis ∂_i of $T_{x_0}M$, and let e^i be the dual frame of T^*M .

It follows that in this trivialization the bundle $\text{End}_{C(M)}\mathcal{F}$, restricted to U , is the trivial bundle with fibre $\text{End}_{C(V)}\mathcal{F} = \text{End}(W)$ (cf. [1, p. 154]).

Let $p_u(x, x_0; t)$ be the smooth kernel of the operator I_t . We transfer the kernel to the neighborhood U of $0 \in V$, thinking of it as taking values in $\text{End}(F) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$, by writing

$$p_u(\mathbf{x}; t) = \tau(x_0, x)p_u(x, x_0; t), \quad \text{where } x = \exp_{x_0}\mathbf{x}.$$

Set

$$\widehat{p}_u(\mathbf{x}; t) = \sigma(p_u(\mathbf{x}; t)), \quad \text{where } \sigma \text{ is the full symbol map.}$$

Then $\widehat{p}_u(\mathbf{x}; t)$ is a $\Lambda(V^*) \otimes \text{End}(W) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$ -valued function on U . Consider the space $\Lambda(V^*) \otimes \text{End}(W) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$ as a $C(V^*) \otimes \text{End}(W)$ module, where the action of $C(V^*)$ on $\Lambda(V^*)$ is the usual one $c(e^j) = e^j \wedge -i_{e_j}$.

The $\Lambda V^* \otimes \text{End}(W) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$ -valued function $\widehat{p}_u(\mathbf{x}; t)$ satisfies the differential equation

$$(\partial_s + \widehat{I}_t(\mathbf{x}))\widehat{p}_u(\mathbf{x}; t) = 0, \tag{42}$$

where \widehat{I}_t is the local expression of I_t in the above trivialization.

Now we introduce Getzler rescaling (cf. [12] and [1, Section 4.3]).

For any $\alpha \in \mathcal{C}^\infty(\mathbb{R}^+ \times U, \Lambda V^* \otimes \text{End}_{C(V^*)}(F) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i))$, let

$$(\delta_\varepsilon \alpha)(t, \mathbf{x}) = \sum_{i=0}^n \varepsilon^{-i} \alpha(\varepsilon^2 t, \varepsilon \mathbf{x})_{[i]}, \tag{43}$$

where $\alpha_{[i]}$ is the i -form component of α .

Set $\widehat{I}_\varepsilon = \delta_\varepsilon \widehat{I}_t \delta_\varepsilon^{-1}$. Then we have

$$(\partial_s + \delta_\varepsilon \widehat{I}_t \delta_\varepsilon^{-1})(\delta_\varepsilon \widehat{p}_u) = \delta_\varepsilon (\partial_s + \widehat{I}_t) \widehat{p}_u = 0. \tag{44}$$

Definition 3.2 (Compare with [1, Definition 4.18]). *The rescaled heat kernel $r(\varepsilon, s, t, \mathbf{x})$ is defined by*

$$r(\varepsilon, s, t, \mathbf{x}) = \varepsilon^n (\delta_\varepsilon \widehat{p}_u)(t, \mathbf{x}). \tag{45}$$

Fix any $T > 1$. The following results are essentially similar to those proved in [1, Section 4.3].

Lemma 3.3 (Compare with [1, Lemma 4.19]). *There exist $\Lambda V^* \otimes \text{End}(W) \widehat{\otimes} R(\vartheta_1, \dots, \vartheta_i)$ -valued polynomials $\gamma_i(s, t, \mathbf{x})$ on $\mathbb{R}^+ \times \mathbb{R}^+ \times V$, such that for every integer N , the function $r^N(\varepsilon, s, t, \mathbf{x})$ defined by*

$$r^N(\varepsilon, s, t, \mathbf{x}) = q_{st}(\mathbf{x}) \sum_{i=-n-q}^{2N} \varepsilon^i \gamma_i(s, t, \mathbf{x}) \tag{46}$$

approximates $r(\varepsilon, s, t, \mathbf{x})$ in the following sense:

For N large enough and $0 < s, t < T$, there is a constant $C(N, k, \alpha) > 0$ such that

$$\begin{aligned} \|\partial_s^k \partial_x^\alpha (r(\varepsilon, s, t, \mathbf{x}) - r^N(\varepsilon, s, t, \mathbf{x}))\| &< C(N, k, \alpha) \varepsilon^{2N - (k+2)q - n + 2}, \\ \text{for } (s, t, \mathbf{x}) &\in (0, T) \times (0, T) \times U \text{ and } 0 < \varepsilon \leq 1, \end{aligned}$$

where $\|\cdot\|$ is defined by using the usual \mathcal{C}^0 -norm (cf. [1, p. 71]) on \mathcal{C}^0 -sections of $\Lambda V^* \otimes \text{End}(W)$ over $\mathbb{R}_+ \times V$ as in (40).

Furthermore, for any $t \in (0, T)$, we have $\gamma_i(0, t, 0) = 0$ if $i \neq 0$, while $\gamma_0(0, t, 0) = \text{Id}$.

We state a general Mehler formula for a generalized harmonic oscillator.

Let B be an $n \times n$ matrix, L be an $m \times m$ matrix, both with coefficients in the commutative algebra \mathcal{A} , where n, m are any positive integers. The generalized harmonic oscillator is the differential operator acting on $\mathcal{A} \otimes \text{End}(\mathbb{C}^m)$ -valued functions defined by

$$H = - \sum_i \left(\partial_i + \frac{1}{4} B_{ij} x_j \right)^2 + L.$$

Theorem 3.4 [10, Proposition 4.7]. For any $a_0 \in \text{End}(\mathbb{C}^m)$, there exists a unique formal solution $p_u(x, B, L, a_0)$ of the heat equation

$$(\partial_u + H_x) p_u(x, B, L, a_0) = 0 \tag{47}$$

of the form

$$p_u(x) = q_u(x) \sum_{k=0}^\infty u^k \Phi_k(x) \tag{48}$$

such that $\Phi_0(0) = a_0$. The function $p_u(x, B, L, a_0)$ is given by the formula

$$(4\pi u)^{-\frac{n}{2}} \widehat{\mathbf{A}}(u\mathcal{D}) \exp\left(\frac{1}{8} \mathcal{C}_{ij} x_i x_j\right) \exp\left(-\frac{1}{4u} \left(\frac{u\mathcal{D}}{2} \coth \frac{u\mathcal{D}}{2}\right)_{ij} x_i x_j\right) \exp(-uL) a_0,$$

where $\mathcal{C}_{ij} = \frac{1}{2}(B_{ij} + B_{ji})$, $\mathcal{D}_{ij} = \frac{1}{2}(B_{ij} - B_{ji})$ and $\mathcal{D} = (\mathcal{D}_{ij})_{n \times n}$.

4 A proof of Theorem 2.5

4.1 An infinitesimal variation formula for the Cappell-Miller holomorphic torsion

Let $\ell \rightarrow g_\ell^{TM}$ be a smooth family of Kähler metrics on M . Let $*_\ell$ be the complex Hodge star operators associated with the metrics g_ℓ^{TM} acting on $\Omega^{p,*}(M)$. Let D_ℓ be the operators acting on $\Omega^{p,*}(M, E)$ defined as in (15) corresponding to g_ℓ^{TM} and ∇^E . Let $U_\ell = (g_\ell^{TM})^{-1} \frac{\partial}{\partial \ell} g_\ell^{TM} \in \text{End}(TM \otimes_{\mathbb{R}} \mathbb{C})$. Denote by $U_\ell^+ \in \text{End}(T^{(1,0)}M)$ the restriction of U_ℓ to $T^{(1,0)}M$. Let R_ℓ^+ be the curvature of $T^{(1,0)}M$ with the natural connection induced by the Levi-Civita connection ∇_ℓ^{TM} associated with the Kähler metric g_ℓ^{TM} . Let $\tau_{\text{holo}, p, \ell}(M, E)$ be the Cappell-Miller holomorphic torsion defined as in (21) corresponding to g_ℓ^{TM} .

The following infinitesimal variation formula for the Cappell-Miller holomorphic torsion, which is closely related to [6, Theorem 1.18], is obtained in [9].

Theorem 4.1 [9, Lemma 4.2 and Lemma 7.1]. As $t \rightarrow 0^+$, for every $k \in \mathbb{N}$, there is an asymptotic expansion

$$\text{Tr}_s \left[*_\ell^{-1} \frac{\partial *_\ell}{\partial \ell} e^{-\frac{1}{2} D_\ell^2} \right] = \sum_{j=-n}^k M_{j, \ell} t^j + o(t^k). \tag{49}$$

Moreover,

$$\frac{\partial}{\partial \ell} \log \tau_{\text{holo}, p, \ell}(M, E) = -M_{0, \ell}. \tag{50}$$

Let $\{e_j\}_{j=1}^{2n}$ be a local orthonormal frame of TM corresponding to g_ℓ^{TM} , and $\{\omega_j\}_{j=1}^n$ be the frame which is related to $\{e_j\}_{j=1}^{2n}$ as in (1).

By a direct calculation, we easily get the following lemma, of which the case $p = 0$ is proved in [6, Proposition 1.19].

Lemma 4.2. *The following identity holds*

$$*_\ell^{-1} \frac{\partial^{*\ell}}{\partial \ell} = \frac{\sqrt{-1}}{4} \dot{\Theta}(e_i, e_j) c(e_i) c(e_j) + \frac{1}{4} \dot{\Theta}(e_i, J e_i) + \sqrt{-1} \dot{\Theta}(\omega_j, \bar{\omega}_i) \omega^j \wedge i_{\omega_i}. \tag{51}$$

4.2 The small time asymptotics of the supertrace of certain heat kernels

For simplicity, we set

$$Q_\ell = *_\ell^{-1} \frac{\partial^{*\ell}}{\partial \ell}. \tag{52}$$

Take any ℓ , say ℓ_0 . In the whole subsection, we always omit the subscript ℓ_0 if there is no confusion. Then Q splits into two parts:

$$\begin{aligned} Q_1 &= \frac{\sqrt{-1}}{4} \dot{\Theta}(e_i, e_j) c(e_i) c(e_j) + \frac{1}{4} \dot{\Theta}(e_i, J e_i), \\ Q_2 &= \sqrt{-1} \dot{\Theta}(\omega_j, \bar{\omega}_i) \omega^j \wedge i_{\omega_i}. \end{aligned} \tag{53}$$

Proposition 4.3. *As $t \rightarrow 0^+$, for every $k \in \mathbb{N}$, there are asymptotic expansions*

$$\text{Tr}_s[Q_1 e^{-tD^2}] = \sum_{j=-1}^k a_j t^j + o(t^k), \quad \text{Tr}_s[Q_2 e^{-tD^2}] = \sum_{j=0}^k b_j t^j + o(t^k). \tag{54}$$

Moreover, we have

$$a_{-1} = (2\pi\sqrt{-1})^{-n} \int_M \frac{\sqrt{-1}}{2} \dot{\Theta} \cdot \text{Td}(R^+) \cdot \text{Tr}[\exp(-R^\mathcal{E})], \tag{55}$$

$$b_0 = (2\pi\sqrt{-1})^{-n} \int_M \sqrt{-1} \text{Td}(R^+) \cdot \text{Tr}[\dot{\Theta}(\omega_i, \bar{\omega}_j) \omega^i \wedge i_{\omega_j} \exp(-R^\mathcal{E})], \tag{56}$$

where

$$R^\mathcal{E} = (\nabla^\mathcal{E})^2 = -\langle R^+ \omega_i, \bar{\omega}_j \rangle_{g_{\ell_0}^{TM}} \omega^i \wedge i_{\omega_j} + R^E. \tag{57}$$

Proof. The known result on the heat kernel asymptotic expansion on the closed manifold M (cf. [1, Theorem 2.30]) implies the existence of the asymptotic expansion as $t \rightarrow 0$. On the other hand, using the standard local index theory techniques of Getzler (cf. [11, 12], [1, Section 4]), we deduce from (53) that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \text{Tr}_s[t Q_1 e^{-tD^2}] &= (2\pi i)^{-n} \int_M \frac{\sqrt{-1}}{2} \dot{\Theta} \cdot \text{Td}(R^+) \cdot \text{Tr}[\exp(-R^\mathcal{E})], \\ \lim_{t \rightarrow 0^+} \text{Tr}_s[Q_2 e^{-tD^2}] &= (2\pi i)^{-n} \int_M \sqrt{-1} \text{Td}(R^+) \text{Tr}[\dot{\Theta}(\omega_i, \bar{\omega}_j) \omega^i \wedge i_{\omega_j} \exp(-R^\mathcal{E})]. \end{aligned}$$

We complete the proof of Proposition 4.3. □

The remaining part of this subsection is devoted to the calculation of a_0 . We follow [6, Section 1(h)] in spirit, although we will not use probability theory in our final calculation. Instead, using the estimates of heat kernels with parameters deduced in last section, we modify the proof of the local index theorem presented in [1, Chapter 4].

We use the convention as in [5, Section 2(f)]. Let $da, d\bar{a}$ be two odd Grassmann variables. If $\eta \in \Lambda(T^*M \otimes_{\mathbb{R}} \mathbb{C}) \hat{\otimes} \mathbb{C}(da, d\bar{a})$ can be written in the form

$$\eta = \eta_0 + da \eta_1 + d\bar{a} \eta_2 + da d\bar{a} \eta_3, \quad \text{where } \eta_i \in \Lambda(T^*M \otimes_{\mathbb{R}} \mathbb{C}), \quad 0 \leq i \leq 3,$$

then we set

$$(\eta)^{\text{dad}\bar{a}} = \eta_3. \tag{58}$$

The following two propositions are proved in [6].

Proposition 4.4 [6, Theorem 1.20]. *The following identity holds,*

$$\frac{\partial}{\partial t}(t\text{Tr}_s[Q_1 \exp(-tD^2)]) = \left(\text{Tr}_s \left[\exp \left(-tD^2 - \sqrt{\frac{t}{2}} daD - \sqrt{\frac{t}{2}} d\bar{a}[D, Q_1] + \text{dad}\bar{a} Q_1 \right) \right] \right)^{\text{dad}\bar{a}}. \tag{59}$$

Proposition 4.5 [6, Theorem 1.21]. *The following identity holds*

$$\begin{aligned} & -tD^2 - \sqrt{\frac{t}{2}} daD - \sqrt{\frac{t}{2}} d\bar{a}[D, Q_1] + \text{dad}\bar{a} Q_1 \\ &= t \left(\nabla_{e_i}^{\text{Cl}\otimes\mathcal{E}} - \frac{1}{2\sqrt{2t}} \text{dac}(e_i) - \frac{d\bar{a}}{2\sqrt{2t}} \sqrt{-1} \dot{\Theta}(e_k, e_i) c(e_k) \right)^2 \\ &+ \frac{\text{dad}\bar{a}}{4} \dot{\Theta}(e_j, J e_j) - \sqrt{\frac{t}{2}} d\bar{a} \frac{c(e_i)}{4} (\nabla_{e_i}^{TM} \dot{\Theta})(e_j, J e_j) - \frac{tK}{4} - \frac{t}{2} c(e_i) c(e_j) F^{\mathcal{E}/S}(e_i, e_j), \end{aligned} \tag{60}$$

where K is the scalar curvature of M , and $F^{\mathcal{E}/S}$ denotes the twisting curvature of the Clifford module $\Lambda(T^{*(0,1)}M) \otimes \mathcal{E}$ given by (cf. [1, p. 117, 148])

$$F^{\mathcal{E}/S} = \frac{1}{2} \text{Tr}_{T^{(1,0)}M} [R^+] + R^{\mathcal{E}}. \tag{61}$$

In (60), we also use the same notations as in [2, Section 3(b)] (compare with (29)).

Set

$$I_t = tD^2 + \sqrt{\frac{t}{2}} daD + \sqrt{\frac{t}{2}} d\bar{a}[D, Q_1] - \text{dad}\bar{a} Q_1. \tag{62}$$

For the bundle $\Lambda(T^{*(0,1)}M) \otimes \mathcal{E}$, by Proposition 4.5, we see that I_t is in the type of operators we study in the last section, and all the results we get there can be applied to the current case.

Lemma 4.6. *The following identity holds*

$$\text{Tr}_s[\exp(-I_t)] = \int_M \text{Tr}_s[k(x, x, 1, t)] dv_M(x), \tag{63}$$

where $k(x, y, s, t)$ is the unique solution of (30).

We trivialize the bundle $\Lambda(T^{*(0,1)}M) \otimes \mathcal{E}$ by parallel transports along the radical geodesics with respect to the Clifford connection $\nabla^{\text{Cl}\otimes\mathcal{E}}$ as in last section. Using [1, Lemmas 4.14 and 4.15] and Proposition 4.5, we can easily deduce the local expression of I_t .

Lemma 4.7. *In the chosen trivialization, the operator I_t , when restricted to U , can be specified as the following operator \hat{I}_t , which is a differential operator on U with coefficients in $C(V^*) \otimes \text{End}(W) \otimes R(da, d\bar{a})$,*

$$\begin{aligned} \hat{I}_t = & -tg^{ij}(\mathbf{x}) \left(\nabla_{\partial_i}^{\text{Cl}\otimes\mathcal{E}} - \frac{da}{2\sqrt{2t}} \langle \partial_i, e_k \rangle(\mathbf{x}) c^k - \frac{\sqrt{-1}}{2\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_k, \partial_i)(\mathbf{x}) c^k \right) \\ & \cdot \left(\nabla_{\partial_j}^{\text{Cl}\otimes\mathcal{E}} - \frac{da}{2\sqrt{2t}} \langle \partial_j, e_l \rangle(\mathbf{x}) c^l - \frac{\sqrt{-1}}{2\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_l, \partial_j)(\mathbf{x}) c^l \right) \\ & + tg^{jj}(\mathbf{x}) \Gamma_{ij}^k(\mathbf{x}) \left(\nabla_{\partial_k}^{\text{Cl}\otimes\mathcal{E}} - \frac{da}{2\sqrt{2t}} \langle \partial_k, e_l \rangle(\mathbf{x}) c^l - \frac{\sqrt{-1}}{2\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_l, \partial_k)(\mathbf{x}) c^l \right) \\ & - \frac{1}{4} \text{dad}\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(\mathbf{x}) + \frac{1}{4} \sqrt{\frac{t}{2}} d\bar{a} \cdot (\nabla_{e_i}^{TM} \dot{\Theta})(e_j, J e_j)(\mathbf{x}) c^i \\ & + \frac{tK}{4}(\mathbf{x}) + \frac{t}{2} F^{\mathcal{E}/S}(e_i, e_j)(\mathbf{x}) c^i c^j. \end{aligned} \tag{64}$$

We compute the Getzler rescaling (cf. (43)) of the operator \widehat{I}_t .

$$\begin{aligned} \widehat{I}_\varepsilon = & -\varepsilon^2 t g^{ij}(\varepsilon \mathbf{x}) \left(\delta_\varepsilon \nabla_{\partial_i}^{\text{Cl}\otimes\mathcal{E}} \delta_\varepsilon^{-1} - \frac{\varepsilon^{-1}}{2\sqrt{2t}} da \cdot \langle \partial_i, e_k \rangle(\varepsilon \mathbf{x}) \cdot (\varepsilon^{-1} e^k \wedge -\varepsilon i_{e_k}) \right. \\ & \left. - \frac{\sqrt{-1}\varepsilon^{-1}}{2\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_k, \partial_i)(\varepsilon \mathbf{x}) \cdot (\varepsilon^{-1} e^k \wedge -\varepsilon i_{e_k}) \right) \\ & \cdot \left(\delta_\varepsilon \nabla_{\partial_j}^{\text{Cl}\otimes\mathcal{E}} \delta_\varepsilon^{-1} - \frac{\varepsilon^{-1}}{2\sqrt{2t}} da \cdot \langle \partial_j, e_l \rangle(\varepsilon \mathbf{x}) \cdot (\varepsilon^{-1} e^l \wedge -\varepsilon i_{e_l}) \right. \\ & \left. - \frac{\sqrt{-1}\varepsilon^{-1}}{2\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_l, \partial_j)(\varepsilon \mathbf{x}) \cdot (\varepsilon^{-1} e^l \wedge -\varepsilon i_{e_l}) \right) \\ & + \varepsilon^2 t g^{ij}(\varepsilon \mathbf{x}) \Gamma_{ij}^k(\varepsilon \mathbf{x}) \left(\delta_\varepsilon \nabla_{\partial_k}^{\text{Cl}\otimes\mathcal{E}} \delta_\varepsilon^{-1} - \frac{\varepsilon^{-1}}{2\sqrt{2t}} da \cdot \langle \partial_k, e_l \rangle(\varepsilon \mathbf{x}) (\varepsilon^{-1} e^l \wedge -\varepsilon i_{e_l}) \right. \\ & \left. - \frac{\sqrt{-1}\varepsilon^{-1}}{2\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_l, \partial_k)(\varepsilon \mathbf{x}) \cdot (\varepsilon^{-1} e^l \wedge -\varepsilon i_{e_l}) \right) \\ & - \frac{1}{4} da d\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(\varepsilon \mathbf{x}) + \frac{\varepsilon}{4} \sqrt{\frac{t}{2}} d\bar{a} \cdot (\nabla_{e_i}^{TM} \dot{\Theta})(e_j, J e_j)(\varepsilon \mathbf{x}) \cdot (\varepsilon^{-1} e^i \wedge -\varepsilon i_{e_i}) \\ & + \frac{\varepsilon^2 t K}{4}(\varepsilon \mathbf{x}) + \frac{\varepsilon^2 t}{2} F^{\mathcal{E}/S}(e_i, e_j)(\varepsilon \mathbf{x}) (\varepsilon^{-1} e^i \wedge -\varepsilon i_{e_i}) (\varepsilon^{-1} e^j \wedge -\varepsilon i_{e_j}), \end{aligned} \tag{65}$$

where (cf. [1, Lemma 4.15])

$$\begin{aligned} \delta_\varepsilon \nabla_{\partial_i}^{\text{Cl}\otimes\mathcal{E}} \delta_\varepsilon^{-1} = & \varepsilon^{-1} \partial_i + \frac{\varepsilon}{8} R_{klij} \mathbf{x}^j (\varepsilon^{-1} e^k \wedge -\varepsilon i_{e_k}) (\varepsilon^{-1} e^l \wedge -\varepsilon i_{e_l}) \\ & + \frac{1}{2} f_{ikl}(\varepsilon \mathbf{x}) (\varepsilon^{-1} e^k \wedge -\varepsilon i_{e_k}) (\varepsilon^{-1} e^l \wedge -\varepsilon i_{e_l}) + g_i(\varepsilon \mathbf{x}), \end{aligned} \tag{66}$$

$R_{klij} = (R(\partial_i, \partial_j)\partial_l, \partial_k)_{x_0}$ is the Riemannian curvature at x_0 , and $f_{ikl}(\mathbf{x}) = O(|\mathbf{x}|^2) \in C^\infty(U)$, $g_i(\mathbf{x}) = O(|\mathbf{x}|) \in \Gamma(U, \text{End}(W))$.

The singular term of \widehat{I}_ε , as $\varepsilon \rightarrow 0$, is that

$$\begin{aligned} & t\varepsilon^{-1} \left(\partial_i + R_{klij} \mathbf{x}^j e^k \wedge e^l \wedge + \frac{1}{2} \varepsilon^{-2} f_{ikl}(\varepsilon \mathbf{x}) e^k \wedge e^l \wedge \right) \\ & \cdot \left(\frac{da}{\sqrt{2t}} \cdot \langle \partial_i, e_k \rangle(0) e^k \wedge + \frac{\sqrt{-1}}{\sqrt{2t}} d\bar{a} \cdot \dot{\Theta}(e_k, \partial_i)(0) e^k \wedge \right) \\ & + t\varepsilon^{-2} \left(\frac{\sqrt{-1}}{4t} da d\bar{a} \cdot \langle \partial_i, e_k \rangle(0) \cdot \dot{\Theta}(e_l, \partial_i)(0) e^k \wedge e^l \wedge \right). \end{aligned} \tag{67}$$

To remove it, we proceed as in [10] by conjugation. Take

$$A(\mathbf{x}, \varepsilon) = \sum_{i,k} \left(\frac{\varepsilon^{-1}}{2\sqrt{2t}} da \langle \partial_i, e_k \rangle(0) \mathbf{x}^i e^k \wedge + \frac{\sqrt{-1}}{2\sqrt{2t}} \varepsilon^{-1} d\bar{a} \cdot \dot{\Theta}(e_k, \partial_i)(0) \mathbf{x}^i e^k \wedge \right),$$

and set $h = \exp(-A)$. Clearly, h is a polynomial of \mathbf{x} and $h(0) = 1$.

Set $\widehat{J}_\varepsilon = h \widehat{I}_\varepsilon h^{-1}$. From (44) and (45), we get

$$(\partial_s + \widehat{J}_\varepsilon)(h(\mathbf{x}) \cdot r(\varepsilon, s, t, \mathbf{x})) = h(\mathbf{x}) \cdot (\partial_s + \widehat{I}_\varepsilon)r(\varepsilon, s, t, \mathbf{x}) = 0. \tag{68}$$

Calculating directly, we deduce

$$\begin{aligned} \widehat{J}_\varepsilon = & -t \left(\partial_i + \frac{1}{8} R_{klij} \mathbf{x}^j e^k \wedge e^l - \frac{\varepsilon^{-1}}{2\sqrt{2t}} da (\langle \partial_i, e_k \rangle(\varepsilon \mathbf{x}) - \langle \partial_i, e_k \rangle(0)) e^k \wedge \right. \\ & \left. - \frac{\sqrt{-1}\varepsilon^{-1}}{2\sqrt{2t}} d\bar{a} \cdot (\dot{\Theta}(e_k, \partial_i)(\varepsilon \mathbf{x}) - \dot{\Theta}(e_k, \partial_i)(0)) e^k \wedge \right) \end{aligned}$$

$$\begin{aligned}
 & -\frac{\sqrt{-1}}{4t}dad\bar{a} \cdot \dot{\Theta}(\partial_i, \partial_j)(0)\mathbf{x}^j \Big)^2 \\
 & -\frac{1}{4}dad\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(0) + \frac{1}{4}\sqrt{\frac{t}{2}}d\bar{a} \cdot (\nabla_{e_i}^{TM}\dot{\Theta})(e_j, J e_j)(0)e^i \wedge \\
 & + \frac{t}{2}F^{\mathcal{E}/S}(e_i, e_j)(0)e^i \wedge e^j + \text{error}(\mathbf{x}, \varepsilon).
 \end{aligned} \tag{69}$$

The symbol $\text{error}(\mathbf{x}, \varepsilon)$ denotes the terms which vanish when $\varepsilon \rightarrow 0$ and will not contribute in the final analysis.

Clearly, for fixed $\mathbf{x} \in U$, $\lim_{\varepsilon \rightarrow 0^+} \widehat{J}_\varepsilon$ exists and is given by

$$\widehat{J}_0 = -t \sum_i \left(\partial_i + \frac{1}{4}B_{ij}\mathbf{x}^j \right)^2 + tL, \tag{70}$$

where

$$B_{ij} = \frac{1}{2}R_{kl ij}e^k \wedge e^l - \sqrt{\frac{2}{t}}\sqrt{-1}d\bar{a} \cdot (\nabla_{\partial_j}^{TM}\dot{\Theta})(e_k, \partial_i)(0)e^k \wedge -\frac{\sqrt{-1}}{t}dad\bar{a} \cdot \dot{\Theta}(\partial_i, \partial_j)(0), \tag{71}$$

$$tL = -\frac{1}{4}dad\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(0) + \frac{1}{4}\sqrt{\frac{t}{2}}d\bar{a} \cdot (\nabla_{e_i}^{TM}\dot{\Theta})(e_j, J e_j)(0)e^i \wedge + \frac{t}{2}F^{\mathcal{E}/S}(e_i, e_j)(0)e^i \wedge e^j. \tag{72}$$

In the above calculation of $\lim_{\varepsilon \rightarrow 0^+} \widehat{J}_\varepsilon$, we use [1, Proposition 1.28], which claims that as $\varepsilon \rightarrow 0^+$, $\langle \partial_i, e_k \rangle(\varepsilon \mathbf{x}) = O(\varepsilon^2)$. Therefore, the term containing da in \widehat{J}_ε vanishes as $\varepsilon \rightarrow 0^+$.

Using the fact that $\widehat{J}_\varepsilon = \widehat{J}_0 + O(\varepsilon)$, we can now show that there are no poles in the Laurent series expansion in ε of $h(\mathbf{x})r(\varepsilon, s, t, \mathbf{x})$. By Lemma 3.3, we have

$$h(\mathbf{x})r(\varepsilon, s, t, \mathbf{x}) \sim q_{st}(\mathbf{x}) \sum_{i=-2n-2}^{\infty} \varepsilon^i h(\mathbf{x})\gamma_i(s, t, \mathbf{x}). \tag{73}$$

We expand the equation (68)

$$(\partial_s + \widehat{J}_\varepsilon)(h(\mathbf{x}) \cdot r(\varepsilon, s, t, \mathbf{x})) = 0$$

in a Laurent series in ε . Lemma 3.3 implies that the leading term

$$q_{st}(\mathbf{x})\varepsilon^{-l}h(\mathbf{x})\gamma_{-l}(s, t, \mathbf{x})$$

of the asymptotic expansion of $h(\mathbf{x})r(\varepsilon, s, t, \mathbf{x})$ satisfies the heat equation

$$(\partial_s + \widehat{J}_0)(q_{st}(\mathbf{x})h(\mathbf{x})\gamma_{-l}(s, t, \mathbf{x})) = 0, \quad \text{for any fixed small } t > 0, \tag{74}$$

which is equivalent to

$$\left(\partial_u + \frac{1}{t}\widehat{J}_0 \right) (q_{st}(\mathbf{x})h(\mathbf{x})\gamma_{-l}(s, t, \mathbf{x})) = 0, \quad \text{where } u = st. \tag{75}$$

Since $\frac{1}{t}\widehat{J}_0$ is a harmonic oscillator, we can apply the generalized Mehler formula Theorem 3.4 to (75). The boundary condition $\gamma_{-l}(0, t, 0) = 0$ for $l > 0$ implies $\gamma_{-l}(s, t, \mathbf{x}) \equiv 0$ for $l > 0$. In particular, we see that there are no poles in the Laurent series expansion of $h(\mathbf{x})r(\varepsilon, s, t, \mathbf{x})$ in powers of ε .

The other thing that we learn from the above argument is that the leading term of the expansion of $h(\mathbf{x})r(\varepsilon, s, t, \mathbf{x})$, i.e. $h(\mathbf{x})r(0, s, t, \mathbf{x}) = q_{st}(\mathbf{x})h(\mathbf{x})\gamma_0(s, t, \mathbf{x})$, satisfies the equation

$$\left(\partial_u + \frac{1}{t}\widehat{J}_0 \right) (q_{st}(\mathbf{x})h(\mathbf{x})\gamma_0(s, t, \mathbf{x})) = 0, \quad \text{with } \gamma_0(0, t, 0) = 1. \tag{76}$$

Using Theorem 3.4, $q_{st}(\mathbf{x})h(\mathbf{x})\gamma_0(s, t, \mathbf{x})$ is given by the explicit formula

$$(4\pi u)^{-n}\widehat{A}(uD) \exp\left(\frac{1}{8}C_{ij}\mathbf{x}^i\mathbf{x}^j\right) \exp\left(-uL - \frac{1}{4u}\left(\frac{uD}{2} \coth \frac{uD}{2}\right)_{ij} \mathbf{x}^i\mathbf{x}^j\right).$$

In particular, we get for any fixed small $t > 0$ that

$$\lim_{\varepsilon \rightarrow 0^+} r(\varepsilon, 1, t, 0) = r(0, 1, t, 0) = (4\pi t)^{-n} \widehat{A}(t\mathcal{D}) \exp(-tL). \tag{77}$$

By (43) and (45), we have

$$r(\varepsilon, s, t, \mathbf{x}) = \sum_{i=0}^{2n} \varepsilon^{2n-i} \widehat{p}_{\varepsilon^2 t s}(\varepsilon \mathbf{X}; \varepsilon^2 t)_{[i]},$$

from which, we obtain

$$r(\varepsilon, s, t, \mathbf{x})|_{s=1, \mathbf{x}=0} = \sum_{i=0}^{2n} \varepsilon^{2n-i} \widehat{p}_{\varepsilon^2 t}(0; \varepsilon^2 t)_{[i]}, \quad (r(\varepsilon, 1, t, 0))_{[2n]} = \widehat{p}_{\varepsilon^2 t}(0; \varepsilon^2 t)_{[2n]}. \tag{78}$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0^+} \text{Tr}[\widehat{p}_{\varepsilon^2 t}(0; \varepsilon^2 t)_{[2n]}] \text{ exists.}$$

Using [1, Proposition 3.21], we get for any fixed small $t > 0$ that

$$\text{Tr}_s[k(x_0, x_0, 1, t)] dv_M(x_0) = (-2\sqrt{-1})^n \text{Tr}[\widehat{p}_t(0; t)_{[2n]}]. \tag{79}$$

Since

$$\lim_{t \rightarrow 0^+} \text{Tr}[\widehat{p}_t(0; t)_{[2n]}] = \lim_{\varepsilon \rightarrow 0^+} \text{Tr}[\widehat{p}_{\varepsilon^2 t}(0; \varepsilon^2 t)_{[2n]}],$$

from (77), (78) and (79), we get

$$\lim_{t \rightarrow 0^+} \text{Tr}_s[k(x_0, x_0, 1, t)] dv_M(x_0) = (2\pi\sqrt{-1}t)^{-n} \text{Tr}[\widehat{A}(t\mathcal{D}) \exp(-tL)]_{[2n]}. \tag{80}$$

Moreover, by Lemma 4.6, we deduce

$$\lim_{t \rightarrow 0} \text{Tr}_s[e^{-It}] = (2\pi\sqrt{-1}t)^{-\frac{n}{2}} \int_M \text{Tr}[\widehat{A}(t\mathcal{D}) \exp(-tL)]. \tag{81}$$

From (61), (71), (72) and (81), we have

$$\begin{aligned} \lim_{t \rightarrow 0} (\text{Tr}_s[e^{-It}])^{\text{dad}\bar{a}} &= (2\pi\sqrt{-1}t)^{-n} \left(\int_M \text{Tr}[\widehat{A}(t\mathcal{D}) \exp(-tL)] \right)^{\text{dad}\bar{a}} \\ &= (2\pi\sqrt{-1})^{-n} \left(\int_M \text{Tr} \left[\widehat{A} \left(\frac{1}{2} R_{kl ij} e^k \wedge e^l - \sqrt{-1} \text{dad}\bar{a} \cdot \dot{\Theta}(\partial_i, \partial_j)(0) \right) \right. \right. \\ &\quad \left. \left. \cdot \exp \left(\frac{1}{4} \text{dad}\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(0) - \frac{1}{2} F^{\mathcal{E}/S}(e_i, e_j)(0) e^i \wedge e^j \right) \right] \right)^{\text{dad}\bar{a}} \\ &= (2\pi\sqrt{-1})^{-n} \int_M \left(\widehat{A} \left(\frac{1}{2} R_{kl ij} e^k \wedge e^l - \sqrt{-1} \text{dad}\bar{a} \cdot \dot{\Theta}(\partial_i, \partial_j)(0) \right) \right. \\ &\quad \left. \cdot \exp \left(\frac{1}{4} \text{dad}\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(0) - \frac{1}{2} \text{Tr}_{T(1,0)M}[R^+](0) \right) \right)^{\text{dad}\bar{a}} \cdot \text{Tr}[\exp(-R^{\mathcal{E}})]. \end{aligned} \tag{82}$$

The facts that $\dot{\Theta}(e_i, J e_i) = 2\text{Tr}_{T(1,0)M}[U^+]$, $\frac{1}{2} R_{ijkl} e^k \wedge e^l = -(R\partial_i, \partial_j)(0)$, and $\dot{\Theta}(\partial_i, \partial_j) = \langle U J \partial_i, \partial_j \rangle$ imply that

$$\begin{aligned} &\widehat{A} \left(\frac{1}{2} R_{kl ij} e^k \wedge e^l - \sqrt{-1} \text{dad}\bar{a} \cdot \dot{\Theta}(\partial_i, \partial_j)(0) \right) \\ &\quad \cdot \exp \left(\frac{1}{4} \text{dad}\bar{a} \cdot \dot{\Theta}(e_i, J e_i)(0) - \frac{1}{2} \text{Tr}_{T(1,0)M}[R^+](0) \right) \\ &= \text{Td}(R^+ - \text{dad}\bar{a} U^+). \end{aligned} \tag{83}$$

Moreover, we get

$$\lim_{t \rightarrow 0} (\text{Tr}_s[e^{-It}])^{\text{dad}\bar{\alpha}} = -(2\pi\sqrt{-1})^{-n} \int_M \frac{\partial}{\partial b} \Big|_{b=0} \text{Td}(R^+ + bU^+) \cdot \text{Tr}[\exp(-R^\mathcal{E})]. \tag{84}$$

Combining Proposition 4.3, Proposition 4.4, Proposition 4.5, (62) and (84) together, we deduce

Proposition 4.8. *The coefficient a_0 in Proposition 4.3 can be calculated by*

$$a_0 = -(2\pi\sqrt{-1})^{-n} \int_M \frac{\partial}{\partial b} \Big|_{b=0} \text{Td}(R^+ + bU^+) \cdot \text{Tr}[\exp(-R^\mathcal{E})]. \tag{85}$$

4.3 A proof of (25)

Since the space of Kähler metrics on TM is convex, we may assume that $\ell \in \mathbb{R} \rightarrow g_\ell^{TM}$ is a smooth family of Kähler metrics on TM such that $g_0^{TM} = g^{TM}$, $g_1^{TM} = g'^{TM}$.

Observing that the following algebraic identity holds

$$\text{Tr}^{\Omega^{p,*}(M)}[\exp(\langle \mathcal{A}\omega_i, \bar{\omega}_j \rangle \omega^i \wedge i\omega_j)] = \sigma_p(\exp \mathcal{A}), \quad \text{for any } \mathcal{A} \in \text{End}(T^{1,0}M),$$

we get from Propositions 4.3 and 4.8 that

$$M_{0,\ell} = -(2\pi\sqrt{-1})^{-n} \int_M \frac{\partial}{\partial b} \Big|_{b=0} (\text{Td}(R_\ell^+ + bU_\ell^+) \cdot \sigma_p(\exp(R_\ell^+ + bU_\ell^+))) \cdot \text{Tr}[\exp(-R^E)]. \tag{86}$$

From Theorem 4.1 and (86), we get

$$\frac{\partial}{\partial \ell} \log \tau_{\text{holo},p,\ell}(M, E) = (2\pi\sqrt{-1})^{-n} \int_M \frac{\partial}{\partial b} \Big|_{b=0} (\text{Td}_p(R_\ell^+ + bU_\ell^+)) \cdot \text{Tr}[\exp(-R^E)]. \tag{87}$$

By the results of [4, Section (e)], the form

$$\varpi = (2\pi\sqrt{-1})^{-n} \int_0^1 \frac{\partial}{\partial b} \Big|_{b=0} (\text{Td}_p(R_\ell^+ + bU_\ell^+)) d\ell \cdot \text{Tr}[\exp(-R^E)] \tag{88}$$

defines an element in P/P' which depends only on g^{TM} and g'^{TM} . According to [4, Theorems 1.27, 1.29, and Corollary 1.30], the component of degree (n, n) of ϖ represents in P/P' the corresponding component of

$$\widetilde{\text{Td}}_p(T^{(1,0)}M, g^{TM}, g'^{TM}) \cdot \text{ch}(E, \nabla^E).$$

Combining with (87), we deduce that

$$\log \frac{\tau'_{\text{holo},p}}{\tau_{\text{holo},p}} = \int_M \varpi = \int_M \widetilde{\text{Td}}_p(T^{(1,0)}M, g^{TM}, g'^{TM}) \cdot \text{ch}(E, \nabla^E), \tag{89}$$

which is equivalent to (25).

The proof of Theorem 2.5 is completed.

Remark 4.9. The referees informed us that there is another way to calculate the coefficient a_0 in Proposition 4.3 modifying the proof of [14, Theorem 2.13] also without using probability theory. The idea is that one trivialize the total bundle $\Lambda(da, d\bar{\alpha}) \otimes \Lambda^p(T^{*(1,0)}M) \otimes \Lambda(T^{*(0,1)}M) \otimes E$ by using a connection similar to ∇'_u in [14, (2.36)], which involve the Grassmann variables $da, d\bar{\alpha}$. This point is exactly the analogue of family situation [1, Section 10.4]. Then one computes from Proposition 4.5 the curvature of the associated connection as in [14, Lemma 2.1], by [1, Proposition 1.18], one obtains that the rescaled operator has a limit. The above trivialization combined with the argument in [1, Section 10.4] yields a direct proof without using the conjugation argument of Donnelly.

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